Table 2 Separation solutions of the similar compressible laminar boundary-layer equations with mass transfer $\sigma = 1.0, f''(0) = 0, f(0) = 1.0$

g_w	$-\hat{\beta}$	$g'(0)$	δ_{tro}^*	θ_{tr}
0	0.822459	1.075035	1.321262	0.488257
0.2	0.816288	0.871091	1.334238	0.485111
0.6	0.774632	0.444047	1.427348	0.468856
1.0	0.712041	0.0	1.586740	0.450854
2.0	0.555091	−1.144072	2.138149	0.420017

layer with mass transfer is integrated for the particular case of unit Prandtl number ($\sigma = 1$), a surface to stagnation enthalpy ratio $g_w = 2.0$, and a similar mass transfer parameter $f_w = 1.0$, i.e.,

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0 \tag{1}$$

$$g'' + fg' = 0 \tag{2}$$

with boundary conditions

$$f(0) = f_w = 1.0 \tag{3a}$$

$$f'(0) = 0 \tag{3b}$$

$$g(0) = g_w = 2.0 \tag{3c}$$

$$f'(\eta \to \infty) = g(\eta \to \infty) \to 1\dagger \tag{3d}$$

These solutions were obtained using the fourth-order Runge-Kutta Nachtsheim-Swigert technique discussed above. The results which are characteristic of reverse flow solutions are given in Table 3 and Figs. 1 and 2.

Table 3 Reverse flow solutions of the similar compressible laminar boundary-layer equations with mass transfer $\sigma = 1.0, g_w = 2.0, f(0) = 1.0$

$-\hat{\beta}$	$-f''(0)$	$-g'(0)$
0.555091	0	1.144072
0.566712	0.1	1.127216
0.573864	0.3	1.087995
0.553773	0.5	1.037012
0.482721	0.7	0.957077
0.4	0.782546	0.886807
0.3	0.793845	0.807186
0.2	0. 34598	0.722124
0.1	0.594141	0.615030
0.05	0.467890	0.536701
0.025	0.368227	0.478004

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† The notation is that of Reference 7.

Reply by Authors to David F. Rogers

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THE comments by David F. Rogers on Ref. 1 should be of value to those interested in extremely high degrees of accuracy. Our main purpose in developing the solutions presented there was to create the separation maps as displayed in Figs. 1, 2, 5, and 9 of Ref. 1. We were interested, predominantly, in obtaining the value of $\hat{\beta}$ at separation with accuracy adequate for engineering work and employment in local similarity analyses.

One observation made by Rogers should be clarified. With the numerical techniques employed, i.e., successive approximation, coupled with the specific initial guess chosen, $f'(\eta) = 1$,‡ no lower branch solutions can possibly be obtained. Careful inspection of the describing integral equations of Ref. 1 indicates that, with this initial guess, appearance of such solutions is precluded. This is not to say that with successive approximation the lower-branch solutions cannot be obtained; a different initial guess exhibiting lower-branch, reverse flow, behavior would probably be required.

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‡ Notation to the same as in Ref. 1.

Comment on "A Method for Extracting Aerodynamic Coefficients from Free-Flight Data"

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Introduction

IN a recent paper[1] Chapman and Kirk of NASA Ames Research Center developed a technique of finding the parameters of various differential equations from flight data. This method is an iterative one that does not require a closed-form solution of the differential equation and is conceptually much more attractive than older methods that employ fits of the data with combinations of damped sine waves. The Chapman-Kirk method does require lengthy calculations on a large computer, and thus it is desirable that it either can provide better results than the previously developed methods requiring less calculation or can be applied to differential equations that cannot be treated by other methods. In reviewing the paper, we note that only one of their four examples might be a realistic case of this class, i.e., a missile with nonlinear damping and static moments. A quasi-linear technique[2–4] can be used on this example, but Chapman and Kirk did not make a

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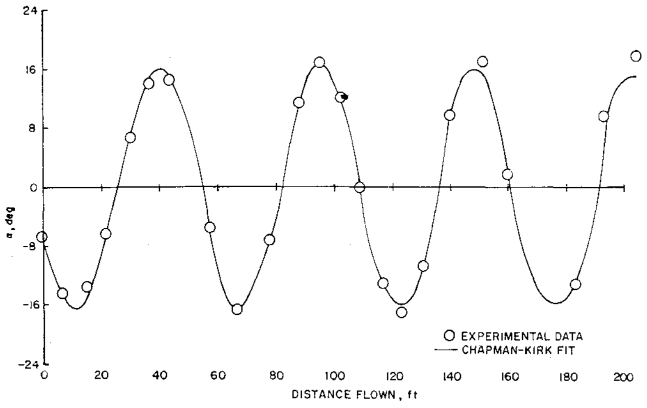

Fig. 1 Pitching motion for run 585.

comparison of their technique with this simpler method. One purpose of this Comment is to make such a comparison.

The second objective of this Comment is to explain a discrepancy in the Chapman-Kirk fit to the motion of Run 585, which is shown in Figure 1. It should be noted that the experimental points for the peaks diverge from the fitted curve. The experimental data indicate a mild growth of peaks of the oscillatory motion, whereas the fitted curve predicts a mild decay of the peaks. Since the fitting process involves the coefficients of a nonlinear differential equation, their contributions to the growth or decay of these peaks is difficult to determine. The quasi-linear method fits the data with a damped sine wave and the parameters of the fit are geometrical observable quantities such as frequency, frequency shift, and exponential damping rate. As will be seen, this allows us to diagnose the strange behavior of Run 585 quickly.

Comparison with Exact Data

The differential equation used was

$$\alpha'' + (C_1 + C_2\alpha^2)\alpha' + (C_3 + C_4\alpha^2)\alpha = 0 \tag{1}$$

This equation was used to fit the motion of four ballistic range flights of the Gemini capsule, with the results $C_1 = -0.00607/\text{ft}$, $C_2 = 0.000105/\text{ft (deg)}^2$, $C_3 = 0.02078/\text{ft}^2$, $C_4 = -0.0000375/\text{ft}^2 \text{ (deg)}^2$. Exact data points were generated from these parameters by integrating Eq. (1) for initial conditions appropriate to these flights.[5] The quasi-linear method assumes that the solution to the nonlinear equation can be approximated by

$$\alpha = 2K \sin\theta$$

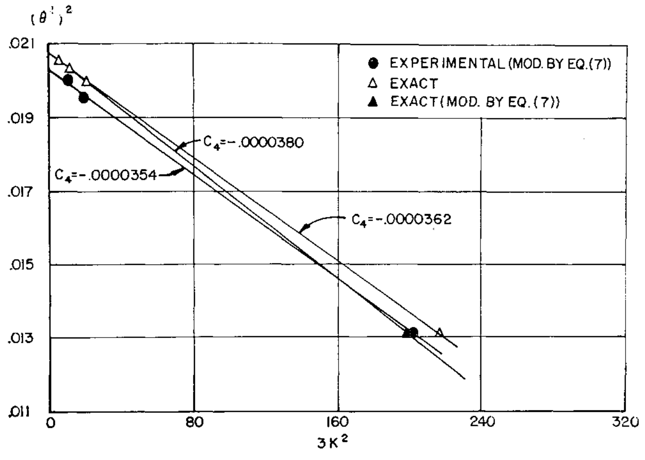

Fig. 2 Quasi-linear frequencies plotted vs amplitude squared.

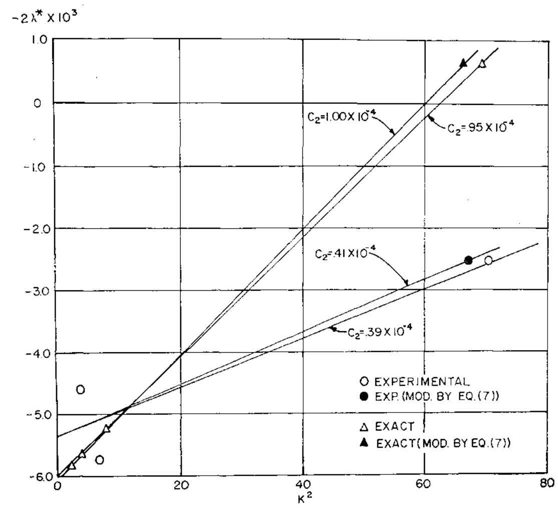

Fig. 3 Constant-frequency damping rates vs amplitude squared.

where

$$K'/K = \lambda = \tfrac{1}{2}\,[C_1 + C_2K^2 - \theta''/\theta']$$
$$= \lambda^* - \theta''/2\theta' \tag{2}$$
$$(\theta')^2 = C_3 + 3C_4K^2$$

Note that if K changes during the flight because of aerodynamic damping the frequency will not be constant but will vary because of its dependence on C_4K^2. We will, therefore, assume θ to be quadratic in x and expand it about the midpoint of the data x_0

$$\theta = \theta_0 + \theta_0'(x - x_0) + (\theta_0''/2)(x - x_0)^2 \tag{3}$$

λ, however, can be approximated by a constant

$$K = K_0 \exp[\lambda(x - x_0)] \tag{4}$$

The results of fitting Eqs. (2–4) are given in Table 1. Equation (2) was also fitted using a constant frequency and it was found that the parameters of Table 1 were unaltered but the standard error of the fit was substantially greater.

The θ_0'''s for the fit of the exact integration are given in Table 2. These can be estimated by differentiating the equation for $(\theta')^2$

$$\theta'' = 3C_4K^2\lambda(\theta')^{-1} \tag{5}$$

The estimates of Eq. (5) are also given in Table 2, and these can be seen to be quite good.

$(\theta')^2$ and $2\lambda^*$ are plotted versus $3K^2$ and K^2 in Figs. 2 and 3. The least squares fits of lines to the points are given in Table 3. The agreement of these values with the exact values is quite good. In comparing the maximum values of α for the exact

Table 1 Quasi-linear parameter

Run no.	2K°	$10^3\lambda$ (1/ft)	θ' (1/ft)	Std. error, °
		Exact integration		
582	2.88	2.94	0.1433	0.004
575	3.81	2.86	0.1427	0.006
584	5.38	2.69	0.1412	0.018
585	16.63	−0.42	0.1148	0.320
		Experimental data		
582[a]	3.09	2.70	0.1486	0.17
575	3.82	2.29	0.1417	0.19
584	5.19	2.93	0.1398	0.14
585	16.82	1.01	0.1149	0.26

[a] Slightly different shape.

Table 2 $10^5\ \theta''$

	Exact		Experimental	
Run No.	Fit	Estimate	Fit	Estimate
582	−0.48	−0.50	0.96	−0.50
575	−0.87	−0.83	0.44	−0.61
584	−1.63	−1.57	−2.02	−1.61
585	2.84	2.85	6.52	−0.69

with those from Eq. (2), however, a systematic overestimate of the exact α_{max} was observed for Run 585, which had the largest amplitude motion. This was felt to be caused by fitting the actual elliptic function solution with a sine wave. This bias can be eliminated through the approximation

$$sn\ u \doteq \sin y(1 + 4q \cos^2 y) \qquad (6)$$

where $q = \exp(-\pi E_1' E_1^{-1})$, $E_1' = E_1(1 - k^2)^{1/2}$, $E_1 = E_1(k)$ complete elliptic integral of first kind, $k = -m(2 + m)^{-1}$, $m = 4K^2C_4C_3^{-1}$, and $u = (2E_1/\pi)y$. If $sn\ u$ is fitted by least squares to $A \sin y$, Eq. (6) yields

$$A = 1 + q \qquad (7)$$

This correction for Run 585 is −2.4%. $3K^2$ and K^2 are each reduced by 4.8% in Figs. 2 and 3 and lines refitted, with the results $C_1 = -0.00602$, $C_2 = 0.000100$, $C_3 = 0.02074$, $C_4 = -0.0000380$. Thus, we see that the quasi-linear technique can yield the nonlinear damping coefficients to 5% and the cubic static moment coefficient to 2% from the exactly calculated points.

Analysis of Experimental Data

The actual flight data can now be fitted by Eqs. (2–4) with the results given in Tables 1 and 2. The standard error of the fits is quite good, but θ'' is now poorly predicted by Eq. (5). This probably means that C_3 and C_4 are not constants but must be allowed to be functions of Mach number,

$$\therefore \theta'' = [6C_4K^2\lambda + C_3' + 3C_4'K^2](2\theta')^{-1} \qquad (8)$$

where $C_j' = (dC_j/dM)(dM/dx)$, M = Mach number. No provision for this effect is made in Chapman-Kirk analysis of these data although it could easily be incorporated by assuming a linear dependence of $C_{3,4}$ on Mach number and thereby increase the number of unknown coefficients of the differential equation to six.

The discrepancy shown in Fig. 1 can now be explained easily. The data contain both varying frequency and exponential damping. The fitted curve implicitly assumes these to be related by Eq. (5). Since the frequency variation dominates, the damping rate is given an erroneous bias. In Fig. 3 the quasi-linear damping is plotted for the experimental data and we see substantially different values of C_1 and C_2 are indicated. These values predict a limit cycle oscillation of amplitude 23° instead of the value of 15° implied by Ref. 1.

Figures 2 and 3 also show an additional advantage of the quasi-linear method. Namely, it gives an indication of the accuracy of the determination of the nonlinear coefficients since they appear as slopes of fitted lines. Indeed, we see that the nonlinear damping coefficient is poorly determined whereas the cubic static moment coefficient is very well determined.

Table 3 Quasi-linear results for exact input

	C_1	C_2	C_3	C_4
Exact	−0.00607	0.000105	0.02078	−0.0000375
Quasi-linear	−0.00600	0.000095	0.02072	−0.0000362
Quasi-linear[a]	−0.00602	0.000100	0.02074	−0.0000380

[a] Modified by Eq. (7).

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Reply by Authors to C. H. Murphy

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THE quasi-linear method in the past has proven to be completely adequate in a great many instances; Dr. Murphy has certainly demonstrated its applicability in this example. However, there are problems for which the quasi-linear method would not be expected to give good results as, for example, when the test configuration departs greatly from being axisymmetric and the nonlinearities in the governing differential equations become large. Further, when using an approximate method on problems of this type, it is difficult to know a priori just how valid the solution is. Theoretical approximations interact in a variety of different ways with errors in the experimental data. The present method[1] eliminates this problem and, conceptually at least, can treat the complete differential equations involving six degrees of freedom.

Reference

[1] Chapman, G. T. and Kirk, D. B., "A Method for Extracting Aerodynamic Coefficients from Free-Flight Data," *AIAA Journal*, Vol. 8, No. 4, April 1970, pp. 753–758.

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Reply by Author to M. N. Rao

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AS pointed out by M. N. Rao in his comments[1] on our earlier studies[2,3] on classes of second- and third-order nonlinear systems, there indeed exists a definite relationship between the various nonlinear terms involved in the governing differential equation which permits it to be reducible to an equivalent linear differential equation (i.e., to an integrable form). However, a more general relationship than that put

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